

To find the y coordinate of the center of mass, we use the second half of formula (4):

$$\bar{y} = \frac{\sum_{i=1}^n m_i \bar{y}_i}{\sum_{i=1}^n m_i} = \frac{\sum_{i=1}^n \rho k_i \Delta x_i (\frac{1}{2} k_i)}{\sum_{i=1}^n \rho k_i \Delta x_i}$$

The denominator is the total mass $\rho \int_a^b f(x) dx$, as before. The numerator is $\frac{1}{2} \rho \sum_{i=1}^n k_i^2 \Delta x_i$, and we recognize $\sum_{i=1}^n k_i^2 \Delta x_i$ as the integral $\int_a^b [f(x)]^2 dx$ of the step function $[f(x)]^2$. Thus,

$$\bar{y} = \frac{\frac{1}{2} \rho \int_a^b [f(x)]^2 dx}{\rho \int_a^b f(x) dx} = \frac{\frac{1}{2} \int_a^b [f(x)]^2 dx}{\int_a^b f(x) dx}.$$

We have derived the formulas for \bar{x} and \bar{y} for the case in which $f(x)$ is a step function; however, they make sense as long as $f(x)$, $xf(x)$, and $[f(x)]^2$ are integrable on $[a, b]$. As usual, we carry over the same formula to general f , so formulas (5) are derived.

Exercises for Section 9.4

- Redo Example 1 by choosing B_1 to consist of m_1 alone and B_2 to consist of m_2 and m_3 .
- Assuming formula (2) and the consolidation principle, derive formula (3) for the case of four masses by dividing the masses into two groups of two masses each.
- Using formulas (2) and (3) for two and three masses, and the consolidation principle, derive formula (3) for four masses.
- Assume that you have derived formula (3) from formula (2) and the consolidation principle for n masses. Now derive formula (3) for $n + 1$ masses.
- Masses of 1, 3, 5, and 7 units are located at the points 7, 3, 5, and 1, respectively, on the x axis. Where is the center of mass?
- Masses of 2, 4, 6, 8, and 10 units are located at the points $x_1 = 0$, $x_2 = 1$, $x_3 = 3$, $x_4 = -1$, and $x_5 = -2$ on the x axis. Locate the center of mass.
- For each integer i from 1 to 100, a point of mass i is located at the point $x = i$. Where is the center of mass? (See Exercise 41(a), Section 4.1.)
- Suppose that n equal masses are located at the points $1, 2, 3, \dots, n$ on a line. Where is their center of mass?

In Exercises 9–12, find the center of mass for the given arrangement of masses.

- 10 grams at (1, 0) and 20 grams at (1, 2).
- 15 grams at (-3, 2) and 30 grams at (4, 2).
- 5 grams at (1, 1), 8 grams at (3, 2), and 10 grams at (0, 0).
- 2 grams at (4, 2), 3 grams at (3, 2), and 4 grams at (5, 3).
- (a) Equal masses are placed at the vertices of an equilateral triangle whose base is the segment from (0, 0) to (1, 0). Where is the center of mass?

(b) The mass at (0, 0) is doubled. Where is the center of mass now?

- Masses of 2, 3, 4, and 5 kilograms are placed at the points (1, 2), (1, 4), (3, 5), and (2, 6), respectively. Where should a mass of 1 kilogram be placed so that the configuration of five masses has its center of mass at the origin?
- Verify the consolidation principle for the situation in which four masses in the plane are divided into two groups containing one mass and three masses each. (Assume that formula (3) holds for $n = 3$.)
- Equal masses are placed at the points (x_1, y_1) , (x_2, y_2) , and (x_3, y_3) . Show that their center of mass is at the intersection point of the medians of the triangle at whose vertices the masses are located.

Find the center of mass of the regions in Exercises 17–22.

- The region under the graph of $4/x^2$ on $[1, 3]$.
- The region under the graph of $1 + x^2 + x^4$ on $[-1, 1]$.
- The region under the graph of $\sqrt{1 - x^2}$ on $[0, 1]$.
- The region under the graph of $\sqrt{1 - x^2/a^2}$ on $[-a, a]$.
- The triangle with vertices at (0, 0), (0, 2), and (4, 0).
- The triangle with vertices at (1, 0), (4, 0), and (2, 3).
- If, in formula (3), we have $a \leq x_i \leq b$ for all x_i , show that $a \leq \bar{x} \leq b$ as well. Interpret this statement geometrically.
- Let a mass m_i be placed at position x_i on a line ($i = 1, \dots, n$). Show that the function $f(x) = \sum_{i=1}^n m_i (x - x_i)^2$ is minimized when x is the center of mass of the n particles.

25. Suppose that masses m_i are located at points x_i on the line and are moving with velocity $v_i = dx_i/dt$ ($i = 1, \dots, n$). The *total momentum* of the particles is defined to be $P = m_1v_1 + m_2v_2 + \dots + m_nv_n$. Show that $P = Mv$, where M is the total mass and v is the velocity of the center of mass (i.e., the rate of change of the position of the center of mass with respect to time).
26. A mass m_i is at position $x_i = f_i(t)$ at time t . Show that if the force on m_i is $F_i(t)$, and $F_1(t) + F_2(t) = 0$, then the center of mass of m_1 and m_2 moves with constant velocity.
27. From a disk of radius 5, a circular hole with radius 2 and center 1 unit from the center of the disk is cut out. Sketch and find the center of mass of the resulting figure.
28. Suppose that $f(x) < g(x)$ for all x in $[a, b]$. Show that the center of mass of the region between the graphs of f and g on $[a, b]$ is located at (\bar{x}, \bar{y}) ,

where

$$\bar{x} = \frac{\int_a^b x[g(x) - f(x)] dx}{\int_a^b [g(x) - f(x)] dx}, \quad \text{and}$$

$$\bar{y} = \frac{\frac{1}{2} \int_a^b [g(x) + f(x)][g(x) - f(x)] dx}{\int_a^b [g(x) - f(x)] dx}.$$

29. Find the center of mass of the region between the graphs of $\sin x$ and $\cos x$ on $[0, \pi/4]$. [Hint: Find the center of mass of each infinitesimal strip making up the region, or use Exercise 28.]
30. Find the center of mass of the region between the graphs of $-x^4$ and x^2 on $[-1, 1]$. (See the hint in Exercise 29.)
- ★31. Find the center of mass of the triangular region with vertices (x_1, y_1) , (x_2, y_2) , and (x_3, y_3) . (For convenience, you may assume that $x_1 \leq x_2 \leq x_3$, $y_1 \leq y_3$, and $y_2 \leq y_3$.) Compare with Exercise 16.

9.5 Energy, Power, and Work

Energy is the integral of power over time, and work is the integral of force over distance.

Energy appears in various forms and can often be converted from one form into another. For instance, a solar cell converts the energy in light into electrical energy; a fusion reactor, in changing atomic structures, transforms nuclear energy into heat energy. Despite the variety of forms in which energy may appear, there is a common unit of measure for all these forms. In the MKS (meter-kilogram-second) system, it is the *joule*, which equals 1 kilogram meter² per second².

Energy is an “extensive” quantity. This means the following: the longer a generator runs, the more electrical energy it produces; the longer a light bulb burns, the more energy it consumes. The rate (with respect to time) at which some form of energy is produced or consumed is called the *power* output or input of the energy conversion device. Power is an *instantaneous* or “intensive” quantity. By the fundamental theorem of calculus, we can compute the total energy transformed between times a and b by integrating the power from a to b .

Power and Energy

Power is the rate of change of energy with respect to time:

$$P = \frac{dE}{dt}.$$

The total energy over a time period is the integral of power with respect to time:

$$E = \int_a^b P dt.$$